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The q -Onsager algebra

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This article gives a summary of the finite-dimensional irreducible representations of the q -Onsager algebra, which are treated in detail in [3].

Fix a nonzero scalar $q \in \mathbb{C}$ which is not a root of unity. Let $\mathcal{A} = \mathcal{A}_q$ denote the associative \mathbb{C} -algebra with 1 defined by generators z, z^* subject to the relations

$$(TD) \begin{cases} [z, z^2 z^* - \beta z z^* z + z^* z^2] &= \delta [z, z^*], \\ [z^*, z^{*2} z - \beta z^* z z^* + z z^{*2}] &= \delta [z^*, z], \end{cases}$$

where $\beta = q^2 + q^{-2}$ and $\delta = -(q^2 - q^{-2})^2$. (TD) can be regarded as a q -analogue of the Dolan-Grady relations and we call \mathcal{A} the q -Onsager algebra. We classify the finite-dimensional irreducible representations of \mathcal{A} . All such representations are explicitly constructed via embeddings of \mathcal{A} into the $U_q(sl_2)$ -loop algebra. As an application, tridiagonal pairs of q -Racah type over \mathbb{C} are classified in the case where q is not a root of unity.

The $U_q(sl_2)$ -loop algebra $\mathcal{L} = U_q(L(sl_2))$ is the associative \mathbb{C} -algebra with 1 generated by $e_i^+, e_i^-, k_i, k_i^{-1}$ ($i = 0, 1$) subject to the relations

$$\begin{aligned} k_0 k_1 &= k_1 k_0 = 1, \\ k_i k_i^{-1} &= k_i^{-1} k_i = 1, \\ k_i e_i^\pm k_i^{-1} &= q^{\pm 2} e_i^\pm, \\ k_i e_j^\pm k_i^{-1} &= q^{\mp 2} e_j^\pm \quad (i \neq j), \\ [e_i^+, e_i^-] &= \frac{k_i - k_i^{-1}}{q - q^{-1}}, \\ [e_i^+, e_j^-] &= 0 \quad (i \neq j), \end{aligned}$$

$$[e_i^\pm, (e_i^\pm)^2 e_j^\pm - (q^2 + q^{-2}) e_i^\pm e_j^\pm e_i^\pm + e_j^\pm (e_i^\pm)^2] = 0 \quad (i \neq j).$$

Note that if we replace $k_0 k_1 = k_1 k_0 = 1$ in the defining relations for \mathcal{L} by $k_0 k_1 = k_1 k_0$, then we have the quantum affine algebra $U_q(\widehat{sl}_2)$: \mathcal{L} is isomorphic to the quotient algebra of $U_q(\widehat{sl}_2)$ by the two-sided ideal generated by $k_0 k_1 - 1$.

Proposition 1 For arbitrary nonzero $s, t \in \mathbb{C}$, there exists an algebra homomorphism $\varphi_{s,t}$ from \mathcal{A} to \mathcal{L} that sends z, z^* to

$$\begin{aligned} z_t(s) &= x(s) + t k(s) + t^{-1} k(s)^{-1}, \\ z_t^*(s) &= y(s) + t^{-1} k(s) + t k(s)^{-1}, \end{aligned}$$

respectively, where

$$\begin{aligned} x(s) &= \alpha(se_0^+ + s^{-1}e_1^-k_1) \quad \text{with } \alpha = -q^{-1}(q - q^{-1})^2, \\ y(s) &= se_0^-k_0 + s^{-1}e_1^+, \\ k(s) &= sk_0. \end{aligned}$$

Moreover $\varphi_{s,t}$ is injective.

We give an overview of finite-dimensional representations of \mathcal{L} that we need to state our explicit construction of irreducible \mathcal{A} -modules via $\varphi_{s,t}$. For $a \in \mathbb{C}$ ($a \neq 0$) and $\ell \in \mathbb{Z}$ ($\ell > 0$), $V(\ell, a)$ denotes the *evaluation module* of \mathcal{L} , i.e., $V(\ell, a)$ is an $(\ell + 1)$ -dimensional vector space over \mathbb{C} with a basis v_0, v_1, \dots, v_ℓ on which \mathcal{L} acts as follows:

$$\begin{aligned} k_0 v_i &= q^{2i-\ell} v_i, \\ k_1 v_i &= q^{\ell-2i} v_i, \\ e_0^+ v_i &= a q [i+1] v_{i+1}, \\ e_0^- v_i &= a^{-1} q^{-1} [\ell - i + 1] v_{i-1}, \\ e_1^+ v_i &= [\ell - i + 1] v_{i-1}, \\ e_1^- v_i &= [i+1] v_{i+1}, \end{aligned}$$

where $v_{-1} = v_{\ell+1} = 0$ and $[j] = [j]_q = (q^j - q^{-j})/(q - q^{-1})$. $V(\ell, a)$ is an irreducible \mathcal{L} -module. We call v_0, v_1, \dots, v_ℓ a *standard basis*.

Let Δ denote the *coproduct* of \mathcal{L} : the algebra homomorphism from \mathcal{L} to $\mathcal{L} \otimes \mathcal{L}$ defined by

$$\begin{aligned} \Delta(k_i^{\pm 1}) &= k_i^{\pm 1} \otimes k_i^{\pm 1}, \\ \Delta(e_i^+) &= k_i \otimes e_i^+ + e_i^+ \otimes 1, \\ \Delta(e_i^- k_i) &= k_i \otimes e_i^- k_i + e_i^- k_i \otimes 1. \end{aligned}$$

Given \mathcal{L} -modules V_1, V_2 , the tensor product $V_1 \otimes V_2$ becomes an \mathcal{L} -module via Δ . Given a set of evaluation modules $V(\ell_i, a_i)$ ($1 \leq i \leq n$) of \mathcal{L} , the tensor product

$$V(\ell_1, a_1) \otimes \dots \otimes V(\ell_n, a_n)$$

makes sense as an \mathcal{L} -module without being affected by the parentheses for the tensor product because of the coassociativity of Δ .

With an evaluation module $V(\ell, a)$ of \mathcal{L} , we associate the set $S(\ell, a)$ of scalars $a q^{-\ell+1}, a q^{-\ell+3}, \dots, a q^{\ell-1}$:

$$S(\ell, a) = \{a q^{2i-\ell+1} \mid 0 \leq i \leq \ell - 1\}.$$

The set $S(\ell, a)$ is called a *q-string* of length ℓ . Two *q-strings* $S(\ell, a), S(\ell', a')$ are said to be *adjacent* if $S(\ell, a) \cup S(\ell', a')$ is a longer *q-string*, i.e., $S(\ell, a) \cup S(\ell', a') = S(\ell'', a'')$ for some ℓ'', a'' with $\ell'' > \max\{\ell, \ell'\}$. It can be easily checked that $S(\ell, a), S(\ell', a')$ are adjacent if and only if $a^{-1}a' = q^{\pm i}$ for some

$$i \in \{|\ell - \ell'| + 2, |\ell - \ell'| + 4, \dots, \ell + \ell'\}.$$

Two *q-strings* $S(\ell, a), S(\ell', a')$ are defined to be *in general position* if they are not adjacent, i.e., if either

- (i) $S(\ell, a) \cup S(\ell', a')$ is not a q -string,
 or
 (ii) $S(\ell, a) \subseteq S(\ell', a')$ or $S(\ell, a) \supseteq S(\ell', a')$.

A multi-set $\{S(\ell_i, a_i)\}_{i=1}^n$ of q -strings is said to be *in general position* if $S(\ell_i, a_i)$ and $S(\ell_j, a_j)$ are in general position for any i, j ($i \neq j$, $1 \leq i \leq n$, $1 \leq j \leq n$). The following fact is well-known and easy to prove. Let Ω be a finite multi-set of nonzero scalars from \mathbb{C} . Then there exists a multi-set $\{S(\ell_i, a_i)\}_{i=1}^n$ of q -strings in general position such that

$$\Omega = \bigcup_{i=1}^n S(\ell_i, a_i)$$

as multi-sets of nonzero scalars. Moreover such a multi-set of q -strings is uniquely determined by Ω .

With a tensor product $V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$ of evaluation modules $V(\ell_i, a_i)$ ($1 \leq i \leq n$), we associate the multi-set $\{S(\ell_i, a_i)\}_{i=1}^n$ of q -strings. The following (i), (ii), (iii) are well-known [1]:

- (i) A tensor product $V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$ of evaluation modules is irreducible as an \mathcal{L} -module if and only if the multi-set $\{S(\ell_i, a_i)\}_{i=1}^n$ of q -strings is in general position.
- (ii) Set $V = V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$, $V' = V(\ell'_1, a'_1) \otimes \cdots \otimes V(\ell'_{n'}, a'_{n'})$ and assume that V, V' are both irreducible as an \mathcal{L} -module. Then V, V' are isomorphic as \mathcal{L} -modules if and only if the multi-sets $\{S(\ell_i, a_i)\}_{i=1}^n, \{S(\ell'_i, a'_i)\}_{i=1}^{n'}$ coincide, i.e., $n = n'$ and $\ell_i = \ell'_i, a_i = a'_i$ for all i ($1 \leq i \leq n$) with a suitable reordering of $S(\ell'_1, a'_1), \dots, S(\ell'_{n'}, a'_{n'})$.
- (iii) Every nontrivial finite-dimensional irreducible \mathcal{L} -module of type (1,1) is isomorphic to some $V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$.

Two multi-sets $\{S(\ell_i, a_i)\}_{i=1}^n, \{S(\ell'_i, a'_i)\}_{i=1}^{n'}$ of q -strings are defined to be *equivalent* if there exists $\varepsilon_i \in \{\pm 1\}$ ($1 \leq i \leq n$) such that $\{S(\ell_i, a_i^{\varepsilon_i})\}_{i=1}^n$ and $\{S(\ell'_i, a'_i)\}_{i=1}^{n'}$ coincide, i.e., $n = n'$ and $\ell_i = \ell'_i, a_i^{\varepsilon_i} = a'_i$ for all i ($1 \leq i \leq n$) with a suitable reordering of $S(\ell'_1, a'_1), \dots, S(\ell'_n, a'_n)$. A multi-set $\{S(\ell_i, a_i)\}_{i=1}^n$ of q -strings is defined to be *strongly in general position* if any multi-set of q -strings equivalent to $\{S(\ell_i, a_i)\}_{i=1}^n$ is in general position, i.e., the multi-set $\{S(\ell_i, a_i^{\varepsilon_i})\}_{i=1}^n$ is in general position for any choice of $\varepsilon_i \in \{\pm 1\}$ ($1 \leq i \leq n$).

Lemma 1 *Let Ω be a finite multi-set of nonzero scalars from \mathbb{C} such that c and c^{-1} appear in Ω in pairs, i.e., c and c^{-1} have the same multiplicity in Ω for each $c \in \Omega$, where we understand that if 1 or -1 appears in Ω , it has even multiplicity. Then there exists a multi-set $\{S(\ell_i, a_i)\}_{i=1}^n$ of q -strings strongly in general position such that*

$$\Omega = \bigcup_{i=1}^n (S(\ell_i, a_i) \cup S(\ell_i, a_i^{-1}))$$

as multi-sets of nonzero scalars. Such a multi-set of q -strings is uniquely determined by Ω up to equivalence.

For an \mathcal{L} -module V , let ρ_V denote the representation of \mathcal{L} afforded by the \mathcal{L} -module V . Then $\rho_V \circ \varphi_{s,t}$ is a representation of \mathcal{A} .

Theorem 1 *The following (i), (ii), (iii) hold.*

(i) *For an \mathcal{L} -module $V = V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$ and nonzero $s, t \in \mathbb{C}$, the representation $\rho_V \circ \varphi_{s,t}$ of \mathcal{A} is irreducible if and only if*

(i.1) *the multi-set $\{S(\ell_i, a_i)\}_{i=1}^n$ of q -strings is strongly in general position,*

(i.2) *none of $-s^2, -t^2$ belongs to $S(\ell_i, a_i) \cup S(\ell_i, a_i^{-1})$ for any i ($1 \leq i \leq n$),*

(i.3) *none of the four scalars $\pm st, \pm st^{-1}$ equals q^i for any $i \in \mathbb{Z}$ ($-d+1 \leq i \leq d-1$).*

(ii) *For \mathcal{L} -modules $V = V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$, $V' = V(\ell'_1, a'_1) \otimes \cdots \otimes V(\ell'_{n'}, a'_{n'})$ and $(s, t), (s', t') \in (\mathbb{C} \setminus \{0\}) \times (\mathbb{C} \setminus \{0\})$, set $\rho = \rho_V \circ \varphi_{s,t}$ and $\rho' = \rho_{V'} \circ \varphi_{s',t'}$. Assume that the representations ρ, ρ' of \mathcal{A} are both irreducible. Then they are isomorphic as representations of \mathcal{A} if and only if the multi-sets $\{S(\ell_i, a_i)\}_{i=1}^n, \{S(\ell'_i, a'_i)\}_{i=1}^{n'}$ are equivalent and*

$$(s', t') \in \{\pm(s, t), \pm(t^{-1}, s^{-1}), \pm(t, s), \pm(s^{-1}, t^{-1})\}.$$

(iii) *Every nontrivial finite-dimensional irreducible representation of \mathcal{A} is isomorphic to $\rho_V \circ \varphi_{s,t}$ for some \mathcal{L} -module $V = V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$ and $(s, t) \in (\mathbb{C} \setminus \{0\}) \times (\mathbb{C} \setminus \{0\})$.*

Let $A, A^* \in \text{End}(V)$ be a TD-pair [2] with eigenspaces $\{V_i\}_{i=0}^d, \{V_i^*\}_{i=0}^d$ respectively. Then we have the split decomposition:

$$V = \bigoplus_{i=0}^d U_i,$$

where

$$U_i = (V_0^* + \cdots + V_i^*) \cap (V_i + \cdots + V_d).$$

By [2, Corollary 5.7], it holds that

$$\dim U_i = \dim V_i = \dim V_i^* \quad (0 \leq i \leq d),$$

and

$$\dim U_i = \dim U_{d-i} \quad (0 \leq i \leq d).$$

Note that $\dim U_i$ is invariant under standardization of A, A^* by affine transformations $\lambda A + \mu I, \lambda^* A^* + \mu^* I$. For an \mathcal{L} -module

$$V = V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n),$$

set $A = \rho_V \circ \varphi_{s,t}(z)$, $A^* = \rho_V \circ \varphi_{s,t}(z^*)$. If the conditions (i.1), (i.2), (i.3) in Theorem 1 hold, then A, A^* are a standardized TD-pair of q -Racah type. Every standardized TD-pair A, A^* of q -Racah type arises in this way. The split decomposition of V for A, A^* coincides with the eigenspace decomposition of the element k_0 of \mathcal{L} acting on V . Thus the generating function for $\dim U_i$

$$g(\lambda) = \sum_{i=0}^d (\dim U_i) \lambda^i$$

is given as follows.

Proposition 2 ([2, Conjecture 13.7])

$$g(\lambda) = \prod_{i=1}^n (1 + \lambda + \lambda^2 + \cdots + \lambda^{\ell_i}).$$

A TD-pair A, A^* is called a *Leonard pair* if $\dim U_i = 1$ for all i ($0 \leq i \leq d$). A standardized TD-pair A, A^* of q -Racah type is a Leonard pair if and only if it is afforded by an evaluation module. In view of this fact, a standardized TD-pair A, A^* of q -Racah type is regarded as a ‘tensor product of Leonard pairs’.

References

- [1] V. Chari and A. Pressley, Quantum affine algebras, *Commun. Math. Phys.* **142** (1991) 261–283.
- [2] T. Ito, K. Tanabe, and P. Terwilliger, Some algebra related to P - and Q -polynomial association schemes, in: *Codes and Association Schemes (Piscataway NJ, 1999)*, Amer. Math. Soc., Providence RI, 2001, 167–192; [arXiv:math.CO/0406556](#).
- [3] T. Ito and P. Terwilliger, The augmented tridiagonal algebra, preprint.

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